J. Fluid Mech. (1991), *vol.* 225, *pp.* 545–555
Printed in Great Britain

Fluctuations in wall-shear stress and pressure at low streamwise wavenumbers in turbulent boundary-layer flow

By D. M. CHASE
Chase Inc., 87 Summer Street, Suite 510, Boston, MA 02110, USA

(Received 19 September 1989 and in revised form 10 October 1990)

Turbulent boundary-layer fluctuations in the incompressive domain are expressed in terms of fluctuating velocity-product 'sources' in order to elucidate relative characteristics of fluctuating wall-shear stress and pressure in the subconvective range of streamwise wavenumbers. Appropriate viscous wall conditions are applied, and results are obtained to lowest order in this Strouhal-scaled wavenumber which serves as the expansion parameter. The spectral amplitudes of pressure and of the shear stress component directed along the wavevector both contain additive terms proportional to source integrals with exponential wall-distance weighting characteristic respectively of the irrotational and the rotational fields. At low wavenumbers, barring unexpected relative smallness of the pertinent boundary-layer source term, the rotational terms become dominant. There the wall pressure and shear-stress component have spectra that approach the same non-vanishing, wavevector-white but generally viscous-scale-dependent level and are totally coherent with phase difference $\frac{1}{2}\pi$. The other, irrotational contributions to the shear-stress and pressure amplitudes likewise bear a simple and previously known, generally wavevector- and frequency-dependent, ratio to one another. In an inviscid limit this contribution to the pressure amplitude reduces to the one obtained previously from inviscid treatments. A representative class of models is introduced for the source spectrum, and the resulting rotational contribution to the spectral density of wall pressure and $\boldsymbol{K}$-aligned shear stress at low (but incompressive) wavenumbers is estimated. It is suggested that this contribution may predominate and account for measured low-wavenumber levels of wall pressure.

1. Introduction

The description of pressure fluctuations on a wall bounded by a turbulent boundary layer has long been a matter of concern in numerous applications. In terms of a spectral density in planar wavevector and frequency, the preponderance of spectral energy of this pressure resides in a 'mean-convective ridge' where the ratio of radian frequency to streamwise wavenumber is of the order of a mean convection speed. Since, however, the spectral transfer function from wall pressure to pertinent response quantities in typical applications provides strong suppression at such relatively high wavenumbers, the level and dependence of wall pressure in the low-wavenumber tail commands particular attention – magnified, moreover, by the related fact that this domain is least amenable to flexible experimental and theoretical investigation.

For the most part, at least until lately, little attention has been given to the spectral density of wall-shear stress. In certain applications, nevertheless, this constitutes a potentially important source similar to wall pressure. For example, where the flow is bounded by a layer of elastomer, fluid-loaded on the outer face, as typically it is, and bonded to a structure on its inner, wall-shear stress is capable of generating not merely shear stress but also normal stress within the layer. The transfer function to this interior normal stress from wall-shear stress, while depending differently on wavenumber and normal distance, is broadly comparable with that from wall pressure. Likewise, there is evidence from applications in cylindrical geometry that low-wavenumber wall-shear stress excites longitudinal waves in the cylindrical wall that drive associated Poisson-coupled pressure waves in an interior fluid, having appreciable consequences for the interior noise field.

A specific feature of fluctuating wall pressure might even suggest that, where low wavenumbers are concerned, it could tend to become weaker than wall-shear stress. According to the Kraichnan–Phillips 'theorem' (Kraichnan 1956; Phillips 1956), as wavenumber decreases below both the Strouhal wavenumber ω/U_∞ (the radian frequency divided by the free-stream speed) and the reciprocal outer scale $\sim \delta^{-1}$ but remains above the sound wavenumber ω/c, c being the sound speed, the spectrum of turbulent wall pressure tends toward zero, probably as K^2 (Chase 1987). There appeared no similar reason, though this point has not been free of controversy, to anticipate that turbulent wall-shear stress at low wavenumbers tends toward zero at all. Such questions are the focus of the present work. Appropriate viscous boundary conditions are applied to the flow in the incompressive domain, paralleling the analysis of pressure by Hariri & Akylas (1985).

According to the principal result obtained, barring vanishing of a certain velocity-product spectrum with vanishing wavenumber, for which no evidence is identified, the spectra of both the wall-shear stress component directed along the wavevector and the wall pressure approach the same non-vanishing, wavevector-white but perhaps viscous-scale-dependent level. On the basis of a model source spectrum, it is suggested that the measured level of low-wavenumber wall pressure may thereby be accountable, as well as the absence of evidence that this behaves in accordance with the Kraichnan–Phillips theorem.

2. Boundary-layer fluctuations in terms of nonlinear sources in the incompressive case

2.1. *Dynamical equations and formal solution*

The flow, regarded here as incompressible, is approximated as streamwise statistically homogeneous with given mean velocity profile $(U(y), 0, 0)$. For the fluctuating velocity field, the dynamic equations are formulated in terms of Fourier–Stieltjes (F–S) amplitudes $(\hat{u}_1, \hat{v}, \hat{u}_3)$, where these are wave-oriented components with $\hat{u}_3$ along the wavevector $\boldsymbol{K}$ (the x_3 direction) in the plane of the wall and $\hat{v}$ normal to the wall. These amplitudes are transforms of the corresponding space–time realization, as given by

$$\hat{u}_j(y, \boldsymbol{K}, \omega) = (2\pi)^{-3} \int \mathrm{d}t \int \mathrm{d}^2\boldsymbol{R} \exp\left[-\mathrm{i}(\boldsymbol{K}\cdot\boldsymbol{R} - \omega t)\right] u_j(y, \boldsymbol{R}, t),$$

where the u_j are components of fluctuating velocity, $\boldsymbol{R}$ is the vector (x, z) in the wall plane and integrals run over the doubly infinite intervals.

The basic equations are taken as the two independent equations for the x_1 and y components of fluctuating vorticity and the continuity equation:

$$\left[\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-K^2\right)+\mathrm{i}(\omega-Uk_x)\right]\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-K^2\right)\hat{v}+\mathrm{i}k_x U''\hat{v} = \mathrm{i}K\hat{S}_1, \tag{1}$$

$$\left[\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-K^2\right)+\mathrm{i}(\omega-Uk_x)\right]\hat{u}_1-\frac{k_z}{K}U'\hat{v} = \mathrm{i}K^{-1}\hat{S}_2, \tag{2}$$

$$\frac{\mathrm{d}\hat{v}}{\mathrm{d}y}+\mathrm{i}K\hat{u}_3 = 0. \tag{3}$$

Here ν is the kinematic viscosity, and k_x and k_z are respectively the streamwise and spanwise components of $\boldsymbol{K}$. $\hat{S}_1$ and $\hat{S}_2$ are the F–S transforms of the x_1 and y components of the vector

$$\begin{aligned}\boldsymbol{S} &= (\nabla\times\boldsymbol{u})\cdot\nabla\boldsymbol{u}-\boldsymbol{u}\cdot\nabla(\nabla\times\boldsymbol{u}) = \nabla\times[\boldsymbol{u}\times(\nabla\times\boldsymbol{u})] = \nabla\times[\tfrac{1}{2}\nabla u^2-\nabla\cdot\boldsymbol{uu}+(\nabla\cdot\boldsymbol{u})\,\boldsymbol{u}]\\ &= -\nabla\times(\nabla\cdot\boldsymbol{uu}),\end{aligned} \tag{4}$$

where $\boldsymbol{u}$ denotes the fluctuating velocity and the last form results from the continuity equation $\nabla\cdot\boldsymbol{u} = 0$ (equation (3)). Boundary conditions required to be satisfied are

$$\hat{v}(0) = \hat{v}'(0) = \hat{u}_1(0) = 0, \quad |\hat{v}(\infty)| < \infty, \quad |\hat{u}_1(\infty)| < \infty,$$

where the arguments are understood as y.

Equation (1) is the inhomogeneous generalization of the familiar Orr–Sommerfeld equation to include the nonlinear terms of the underlying Navier–Stokes equations, regarded as given sources. Equation (2) is the corresponding equation for the velocity amplitude $\hat{u}_1$ along the wavefront.

In terms of the fluctuating velocity components (u_1, u_2, u_3), whose F–S transforms $(\hat{u}_1, \hat{v}, \hat{u}_3)$ appear in (1) to (3), $\hat{S}_1$ and $\hat{S}_2$ in (1) and (2), by use of (4), may be written

$$\hat{S}_1 = -\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}+K^2\right)T_{32}-\mathrm{i}K\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)(T_{33}-T_{22}), \tag{5a}$$

$$\hat{S}_2 = -\mathrm{i}K\left[\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)T_{12}+\mathrm{i}KT_{32}\right], \tag{5b}$$

where $T_{ij}(y, \boldsymbol{K}, \omega) \equiv (u_i u_j)_\wedge$ and a subscript $\wedge$ signifies a F–S transform ($\wedge$ omitted from T_{ij}). (Note that $(u_i u_j)_\wedge$ for $\omega \neq 0$ may be regarded as the transform equivalently of $u_i u_j$ or of $u_i u_j - E(u_i u_j)$.)

By use of the Navier–Stokes equations and the continuity equation, the fluctuating pressure amplitude (^ omitted) may be expressed as

$$p = \rho K^{-2}[\nu(\hat{v}'''-K^2 v')+\mathrm{i}(\omega-Uk_x)\,\hat{v}'+\mathrm{i}k_x U'\hat{v}-K^2T_{33}+\mathrm{i}KT'_{32}] \tag{6}$$

where ρ is the fluid density and a prime denotes $\mathrm{d}/\mathrm{d}y$. In particular, at the (rigid) wall $(y = 0)$

$$p(0) = \rho\nu K^{-2}\hat{v}'''(0). \tag{7}$$

Similarly, the F–S amplitude of the x_3 component of wall-shear stress, i.e. of the component along $\boldsymbol{K}$, by use of the continuity equation (3), becomes

$$\tau_3 = \rho\nu\hat{u}'_3(0) = \mathrm{i}\rho\nu K^{-1}\hat{v}''(0), \tag{8}$$

and the amplitude of the x_1 component is

$$\tau_1 = \rho\nu\hat{u}'_1(0). \tag{9}$$

Define the dimensionless parameters

$$\epsilon \equiv U_\infty k_x/\omega, \quad \gamma_1^2 \equiv \mathrm{i}-\nu K^2/\omega, \quad \gamma_2^2 \equiv -\nu K^2/\omega, \tag{10}$$

and introduce also the scaled variables

$$\zeta \equiv (\omega/\nu)^{\frac{1}{2}} y, \quad V(\zeta) \equiv U((\nu/\omega)^{\frac{1}{2}}\zeta)/U_\infty, \tag{11}$$

the former being wall distance scaled on the characteristic penetration depth for rotational flow and the latter the normalized mean velocity.

Equations (1) and (2) may be rewritten without approximation as

$$\left[\left(\frac{d^2}{d\zeta^2}+\gamma_1^2-i\epsilon V\right)\left(\frac{d^2}{d\zeta^2}+\gamma_2^2\right)+i\epsilon V''\right]\hat{v} = \Sigma_1, \tag{12}$$

$$\left(\frac{d^2}{d\zeta^2}+\gamma_1^2-i\epsilon V\right)\hat{u}_1 = \Sigma_+, \tag{13}$$

where

$$\Sigma_1 = i\nu K\omega^{-2}\hat{S}_1, \tag{14a}$$

$$\Sigma_+ \equiv \omega^{-1}(iK^{-1}\hat{S}_2+k_z K^{-1}U'\hat{v}). \tag{14b}$$

2.2. *Solution to lowest order in streamwise Strouhal-scaled wavenumber*

A perturbation expansion is now introduced with respect to the 'subconvective' parameter ϵ defined in (10) by inserting into (12) and (13) assumed expansions:

$$\left.\begin{aligned}\hat{v} &= v_0+\epsilon v_1+\dots,\\ \hat{u}_1 &= u_{10}+\epsilon u_{11}+\dots.\end{aligned}\right\} \tag{15}$$

Equating parts of equal order in those equations with the inhomogeneous parts Σ_1 and Σ_+ regarded as fully accounted for in v_0 and u_{10} yields

$$\left(\frac{d^2}{d\zeta^2}+\gamma_1^2\right)\left(\frac{d^2}{d\zeta^2}+\gamma_2^2\right)v_0 = \Sigma_1, \tag{16a}$$

$$\left(\frac{d^2}{d\zeta^2}+\gamma_1^2\right)u_{10} = \Sigma_+ \tag{16b}$$

$$\left(\frac{d^2}{d\zeta^2}+\gamma_1^2\right)\left(\frac{d^2}{d\zeta^2}+\gamma_2^2\right)v_1 = i\left(V\left(\frac{d^2}{d\zeta^2}+\gamma_2^2\right)-V''\right)v_0, \tag{17a}$$

$$\left(\frac{d^2}{d\zeta^2}+\gamma_1^2\right)u_{11} = iVu_{10}, \quad \text{etc.} \tag{17b}$$

Let $g_j(\zeta)\,(j=1,\dots,4)$ denote a set of four independent arbitrarily normalized solutions to the homogeneous form of (16a) obtained by replacing Σ_1 by zero. A general solution of (16a) may then be formed according to the method of variation of parameters (Bender & Orszag 1978) as

$$\hat{v}_0(\zeta) = \sum_j g_j(\zeta)\int_0^\zeta ds\,\Sigma_1(s)\,w_j(s)+\sum_j b_j g_j(\zeta), \tag{18}$$

where the sums run from 1 to 4, the argument of v_0 is now regarded as ζ in place of y, and the coefficients b_j are to be established from the boundary conditions. Here $w_j(s) = W_j(s)/W(s)$, where $W(s)$ is the determinant of the Wronskian matrix $\{g_j^{(n)}(s)\}$ $(n = 0,1,2,3)$ and $W_j(s)$ is the cofactor of $g_j'''(s)$. Similarly, let $f_j(\zeta)\,(j=1,2)$ denote a set of two independent solutions to the homogeneous form of (16b). Then a general solution of (16b) may be formed as

$$\hat{u}_{10}(\zeta) = \sum_j f_j(\zeta)\int_0^\zeta ds\,\Sigma_+(s)\,r_j(s)+\sum_j c_j f_j(\zeta), \tag{19}$$

where $j = 1$ and 2 and $r_j(s) = R_j(s)/R(s)$, R being the determinant and R_j the cofactor of $f_j'(s)$ of the Wronskian matrix $\{f_j^{(n)}(s)\}\,(n = 0, 1)$.

Let the phases of γ_1 and γ_2 be chosen such that

$$\left.\begin{aligned}\gamma_1 &= 2^{-\frac{1}{2}}(1+\mathrm{i})\,(1+\mathrm{i}\nu K^2/\omega)^{\frac{1}{2}}, \quad \mathrm{Im}\,(\gamma_1) > 0,\\ \gamma_2 &= \mathrm{i}(\nu/\omega)^{\frac{1}{2}}K.\end{aligned}\right\} \tag{20}$$

One may take as four independent solutions $g_j(\zeta)$ of the homogeneous form of (16a) $g_{\alpha\pm} = \exp(\pm \mathrm{i}\gamma_\alpha \zeta)\,(\alpha = 1, 2)$. One then obtains for the w_j, say $w_{\alpha\pm}$,

$$w_{1\pm}(\zeta) = \pm\mathrm{i}\tfrac{1}{2}\gamma_1^{-1}(\gamma_2^2-\gamma_1^2)^{-1}\exp(\pm\mathrm{i}\gamma_1\zeta), \tag{21}$$

and $w_{2\pm}$ is obtained by exchanging indices 1 and 2 in this expression.

By the prescription above, the solution to (16a) may be written

$$\begin{aligned}v_0(\zeta) = \mathrm{i}\tfrac{1}{2}(\gamma_2^2-\gamma_1^2)^{-1}&\left\{\gamma_1^{-1}\left[\mathrm{e}^{-\mathrm{i}\gamma_1\zeta}\int_0^\zeta \mathrm{d}s\,\Sigma_1(s)\,\mathrm{e}^{\mathrm{i}\gamma_1 s} - \mathrm{e}^{\mathrm{i}\gamma_1\zeta}\int_0^\zeta \mathrm{d}s\,\Sigma_1(s)\,\mathrm{e}^{-\mathrm{i}\gamma_1 s}\right]\right.\\ &\left.-\gamma_2^{-1}\left[\mathrm{e}^{-\mathrm{i}\gamma_2\zeta}\int_0^\zeta \mathrm{d}s\,\Sigma_1(s)\,\mathrm{e}^{\mathrm{i}\gamma_2 s} - \mathrm{e}^{\mathrm{i}\gamma_2\zeta}\int_0^\zeta \mathrm{d}s\,\Sigma_1(s)\,\mathrm{e}^{-\mathrm{i}\gamma_2 s}\right]\right\}\\ &+a_{1+}\,\mathrm{e}^{\mathrm{i}\gamma_1\zeta}+a_{1-}\,\mathrm{e}^{-\mathrm{i}\gamma_1\zeta}+a_{2+}\,\mathrm{e}^{\mathrm{i}\gamma_2\zeta}+a_{2-}\,\mathrm{e}^{-\mathrm{i}\gamma_2\zeta},\end{aligned} \tag{22}$$

where the $a_{\alpha\pm}$ are constants to be determined. The condition $v_0(0) = 0$ implies that

$$a_{1+}+a_{1-}+a_{2+}+a_{2-} = 0, \tag{23a}$$

and the condition $v_0'(0) = 0$ that

$$\gamma_1(a_{1+}-a_{1-})+\gamma_2(a_{2+}-a_{2-}) = 0. \tag{23b}$$

Further, to ensure that $v_0(\zeta)$ remains bounded as $\zeta \to \infty$, in view of (20), the coefficients of $\mathrm{e}^{-\mathrm{i}\gamma_1\zeta}$ and $\mathrm{e}^{-\mathrm{i}\gamma_2\zeta}$ in this limit must vanish, whence

$$\left.\begin{aligned}a_{1-}+\mathrm{i}\tfrac{1}{2}\gamma_1^{-1}(\gamma_2^2-\gamma_1^2)^{-1}N_1 &= 0,\\ (\gamma_1/\gamma_2)\,(a_{1+}-a_{1-})-(a_{1+}+a_{1-})-\mathrm{i}\gamma_2^{-1}(\gamma_2^2-\gamma_1^2)^{-1}N_2 &= 0,\end{aligned}\right\} \tag{23c}$$

where

$$N_\alpha \equiv \int_0^\infty \mathrm{d}s\,\Sigma_1(s)\,\mathrm{e}^{\mathrm{i}\gamma_\alpha s} \quad (\alpha = 1, 2). \tag{24}$$

Equations (23) suffice to determine the $a_{\alpha\pm}$ in terms of the N_α.

By repeated differentiation of (22) and evaluation at $\zeta = 0$, it is found that

$$\rho\nu K^{-2}v_0'''(0) = \mathrm{i}\rho\omega K^{-1}(\gamma_1-\gamma_2)^{-1}\,(N_1-\gamma_1 N_2/\gamma_2), \tag{25a}$$

$$\mathrm{i}\rho\nu K^{-1}v_0''(0) = \rho\omega K^{-1}(\gamma_1-\gamma_2)^{-1}\,(N_1-N_2), \tag{25b}$$

where the primes, as in (7) and (8), now represent differentiation not with respect to ζ but to y (of (11)).

Since the T_{ij} are convolutions of fluctuating velocity amplitudes $\hat{u}_i, \hat{u}_j$ (see below), the required vanishing of the spectral densities of these velocity components at the rigid wall apparently implies that the amplitudes $T_{ij}(0)$, $T_{ij}'(0)$ may, as usual, be regarded as vanishing there. To provide for possible relaxation of this condition, however, the analysis will be formally generalized to the case where no such assumption is made.

To this end, in view of (6), equation (7) is generalized to become

$$p(0) = \rho[\nu K^{-2}v'''(0)-T_{33}(0)+\mathrm{i}K^{-1}T_{32}'(0)]. \tag{26}$$

Likewise, the wall-shear-stress amplitude must be generalized to include a hypothetical residual Reynolds-stress contribution at the wall, so that (8) and (9) are generalized to become

$$\tau_3 = \rho[\mathrm{i}\nu K^{-1}v''(0) - T_{32}(0)], \tag{27}$$

$$\tau_1 = \rho[\nu\hat{u}_1'(0) - T_{12}(0)]. \tag{28}$$

Proceeding, then, set

$$T_{ij}(y, \boldsymbol{K}, \omega) = t_{ij}(\zeta, \boldsymbol{K}, \omega), \tag{29}$$

and let derivatives of T_{ij}, t_{ij} with respect to y, ζ, respectively, be denoted by a prime. (To render t_{ij} and t'_{ij} non-dimensional, it is noted, one would need to include in the right member of (29) a squared-velocity factor, e.g. U_∞^2.) By (5a) and (14a)

$$\Sigma_1 = -\mathrm{i}\nu\omega^{-2}K[(\omega/\nu)\,t''_{23} + \mathrm{i}K(\omega/\nu)^{\frac{1}{2}}(t'_{33}-t'_{22}) + K^2 t_{23}]. \tag{30}$$

Insertion of (30) in (24) and integration by parts yields

$$N_\alpha = \mathrm{i}K\omega^{-1}\Big\{(\gamma_\alpha^2+\gamma_2^2)\int_0^\infty \mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\gamma_\alpha\zeta}t_{32} + \mathrm{i}\gamma_2\gamma_\alpha\int_0^\infty \mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\gamma_\alpha\zeta}(t_{33}-t_{22}) + t'_{32}(0) - \mathrm{i}\gamma_\alpha t_{32}(0) + \gamma_2[t_{33}(0)-t_{22}(0)]\Big\}. \tag{31}$$

To lowest order in ϵ, the amplitudes $p^0(0)$ and τ_3^0 of wall pressure and $\boldsymbol{K}$-aligned shear stress are obtained from (26) and (27) by substitution from (25) and (31), recalling (29):

$$p^0(0) = \rho[\sigma_1+\sigma_2 - T_{22}(0)], \tag{32}$$

$$\tau_3^0 = -\mathrm{i}\rho(\sigma_1+\sigma_2\gamma_2/\gamma_1), \tag{33}$$

where
$$\sigma_1 = -(\gamma_1-\gamma_2)^{-1}\int_0^\infty \mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\gamma_1\zeta}[(\gamma_1^2+\gamma_2^2)\,t_{32} + \mathrm{i}\gamma_1\gamma_2(t_{33}-t_{22})], \tag{34}$$

$$\sigma_2 = \mathrm{i}\gamma_1\gamma_2(\gamma_1-\gamma_2)^{-1}\int_0^\infty \mathrm{d}\zeta\,\mathrm{e}^{\mathrm{i}\gamma_2\zeta}(t_{33}-t_{22}-\mathrm{i}2t_{32}) \tag{35a}$$

$$= -\gamma_1(\gamma_1-\gamma_2)^{-1}K\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-Ky}(T_{33}-T_{22}-\mathrm{i}2T_{32}), \tag{35b}$$

and the arguments of the t_{ij}, T_{ij} are suppressed. Of the 'wall terms' in (32) and (33), only $T_{22}(0)$ in (32) has failed to cancel, and it will hereafter be regarded as vanishing.

Alternative to the present formulation of the equations of motion based on elimination of the pressure amplitude and use of the vorticity equations (1) and (2), one may proceed instead from the coupled equations for the F–S amplitudes $\hat{v}(y)$, $p(y)$ of normal velocity and pressure, as done by Hariri & Akylas (1985) with slight compressibility included. In the incompressive case these become

$$\left[\nu\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-K^2\right)+\mathrm{i}(\omega-Uk_x)\right]\hat{v} - \frac{p'}{\rho} = \hat{T}_2,$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-K^2\right)\frac{p'}{\rho} + \mathrm{i}2k_x U'\hat{v} = -(\boldsymbol{\nabla}\cdot\boldsymbol{T})_\wedge,$$

in which, in terms of the earlier T_{ij},

$$(\boldsymbol{\nabla}\cdot\boldsymbol{T})_\wedge = T''_{22} + \mathrm{i}2KT'_{32} - K^2T_{33},$$
$$\hat{T}_2 = \mathrm{i}KT_{32} + T'_{22}.$$

It has been verified that this formulation yields, as it should, the same results as presented above.

This latter formulation, with the equations for the x and z components $\hat{u}$, $\hat{w}$ of the fluctuating velocity amplitude adjoined to those above, likewise affords the simplest way to derive a result also for the component τ_1^0 of wall-shear-stress amplitude normal to the wavevector $\boldsymbol{K}$. Paralleling (33) to (35) for τ_3^0, one finds

$$\tau_1^0 = \rho \int_0^\infty \mathrm{d}\zeta\, \mathrm{e}^{\mathrm{i}\gamma_1\zeta}(\mathrm{i}\gamma_1 t_{12} - \gamma_2 t_{31}).$$

In this instance there is no contribution of the kind ('irrotational') represented by the second term of (33) for τ_3^0. Together, the results for τ_1^0, τ_3^0 complete the determination of the wall-stress vector, $\boldsymbol{\tau}^0$.

In the application where fluctuating wall-shear stress excites an elastomer layer (fluid-loaded), it is noted, only the component τ_3 along $\boldsymbol{K}$ is capable of producing non-vanishing *normal* components of stress within the elastomer.

3. Further analysis and interpretation

It is noted first that σ_1, σ_2 represent, respectively, rotational and irrotational contributions, since by (20) the exponential factors entering (34) and (35) may be written

$$\left.\begin{aligned} \exp(\mathrm{i}\gamma_2\zeta) &= \exp(-Ky), \\ \exp(\mathrm{i}\gamma_1\zeta) &= \exp[-2^{\frac{1}{2}}(1-\mathrm{i})(\omega/\nu + \mathrm{i}K^2)^{\frac{1}{2}}y]. \end{aligned}\right\} \tag{36}$$

The following discussion of the magnitudes of distinct terms and contributions in (34) and (35) is to be interpreted in terms of the magnitudes of the corresponding *spectra* that are formed in the usual way from the F–S amplitudes. Where pertinent, it is supposed that the t_{ij} in the relevant domains are comparable with one another and that no significant cancellations occur due to coherence between terms, as reflected in their cross-spectra.

Consider first the part σ_2. In the inviscid limit where $\nu K^2/\omega \equiv -\gamma_2^2 \rightarrow 0$, (35*b*) yields in (32)

$$p_2^0 \rightarrow -\rho K \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-Ky} T_s, \quad T_s \equiv T_{33} - \mathrm{i}2T_{32} - T_{22}. \tag{37}$$

This, in fact, is the usual inviscid result in the prescribed limit, given, for example, by Chase & Noiseux (1982), (23) with $\epsilon = 0$. When K is regarded as of the same order as k_x ($k_x/K = O(\epsilon^0)$), contrary to the assumption underlying the present treatment (as well as that of Hariri & Akylas 1985), the present perturbation expansion becomes disordered. A proper treatment in this case, as given originally by Bergeron (1973) and rederived by Chase & Noiseux (1982, equation 24), results to lowest order in ϵ in the appearance of an additional term in T_s in (30) corresponding to multiplication of T_{32} in (37) for T_s by $1 - k_x U'/K\omega$.

More generally than (37), for arbitrary $\nu K^2/\omega$, the contributions due to σ_2 in (32) and (33) may be written as

$$p_2^0 = -\rho \left[1 - \frac{\mathrm{i}K}{(\mathrm{i}\omega/\nu - K^2)^{\frac{1}{2}}}\right]^{-1} K \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-Ky}\, T_s, \tag{38}$$

$$\tau_{32}^0 = \frac{K}{(\mathrm{i}\omega/\nu - K^2)^{\frac{1}{2}}} p_2^0. \tag{39}$$

The relation between the irrotational contributions to wall-shear stress and pressure given by (39) agrees with one given by a treatment by Howe (1989, equation 2.9).

In the low-wavenumber limit $K \to 0$ (but subject here to the condition $K \gg \omega/c$ for incompressibility), one has $\gamma_2 \to 0$. Furthermore, validity of the conditions that underlie the Kraichnan–Phillips theorem imply that in the integral in (37) $T_s \to 0$ uniformly in K as $y \to \infty$, so that p_2^0 as given by (37) vanishes as $K \to 0$. (For a related discussion, see Chase 1987.) Hence, the only term remaining in (34) and (35) that need not vanish is that from the term $\propto \gamma_1^2 t_{32}$ in the integrand of (34). Thus, in this limit (32) and (33) yield the results

$$\frac{\tau_3^0}{\rho} \to \mathrm{i}^{\frac{3}{2}} \int_0^\infty \mathrm{d}\zeta \exp(\mathrm{i}^{\frac{3}{2}}\zeta)\, t_{32} \quad (K \to 0)$$

$$= -2^{-\frac{1}{2}}(1-\mathrm{i})\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}y \exp\left[-2^{-\frac{1}{2}}(1-i)\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} y\right] T_{32}, \tag{40}$$

$$p^0(0) \to \mathrm{i}\tau_3^0 \quad (K \to 0), \tag{41}$$

where arguments $\boldsymbol{K}$, ω and ζ (or y) are still suppressed in t_{32} (or T_{32}).

According to (40) and (41), at the low wavenumbers under discussion here, assuming that t_{32} does not vanish with K, both the wall-shear stress ($\boldsymbol{K}$-oriented component) and wall pressure approach non-vanishing, wavevector-white levels, and these levels, moreover, are equal, the two amplitudes being perfectly coherent and differing by $\frac{1}{2}\pi$ in phase. Since the flow has been treated as incompressible and the conditions for applicability of the Kraichnan–Phillips theorem have not been violated, the possible non-vanishing of the wall-pressure spectrum as $K \to 0$ clearly results from the proper inclusion of the viscous wall condition (cf. also Hariri & Akylas 1985 for pressure in the case of slight compressibility).

In the limit where $k_x/K = O(\epsilon^0)$, the result (40) for the wall-shear stress component τ_3 to lowest order implies also the form for the component in an arbitrary direction in the plane. Specifically, when $K \to 0$ there can be no dependence on $\boldsymbol{K}$, and the stress vector must be given by the covariant generalization

$$\frac{\boldsymbol{\tau}^0}{\rho} \to \mathrm{i}^{\frac{3}{2}} \int_0^\infty \mathrm{d}\zeta \exp(\mathrm{i}^{\frac{3}{2}}\zeta)\, \boldsymbol{t}_2, \tag{42}$$

where $\boldsymbol{t}_2$ denotes the vector with (1, 3) components (t_{22}, t_{32}). In this limit the result has been verified by explicit calculation (omitted here) of the component τ_1 of (9).

For orientation and a guide to possible future modelling consistent with the preceding formal development, the relation is recalled that expresses the velocity-product amplitudes T_{ij} as a convolution of amplitudes $\hat{u}_j$ of velocity components:

$$T_{ij}(y, \boldsymbol{K}, \omega) = \int \mathrm{d}\boldsymbol{K}' \int \mathrm{d}\omega'\, \hat{u}_i(y, \boldsymbol{K}-\boldsymbol{K}', \omega-\omega')\, \hat{u}_j(y, \boldsymbol{K}', \omega'). \tag{43}$$

In particular, T_{32} may be expressed as

$$T_{32}(y, \boldsymbol{K}, \omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega' \int_0^\infty \mathrm{d}K'K' \int_{-\pi}^{\pi} \mathrm{d}\phi'\, \hat{v}(y, \boldsymbol{K}-\boldsymbol{K}', \omega-\omega')$$
$$\times [\hat{u}_{3'}(y, \boldsymbol{K}', \omega')\cos\phi' - \hat{u}_{1'}(y, \boldsymbol{K}', \omega')\sin\phi'], \tag{44}$$

where $\hat{u}_3$ refers to the velocity component along $\boldsymbol{K}'$, $\hat{u}_1$ to the orthogonal component in the plane, and ϕ' is the polar angle of $\boldsymbol{K}'$ measured from $\boldsymbol{K}$. At low wavenumbers $K(\ll \omega/U_\infty)$, T_{ij} may derive its dominant contribution from a domain of $\boldsymbol{K}'$, ω' where one or the other of the two factors in the integrand has its mean-convective ridge (e.g. $k'_x \sim \omega'/U_\infty$). There appears no grounds for expecting that the result of this convolution for T_{32} should vanish as $K \to 0$.

The present paper, it is emphasized, like those of Bergeron (1973) and Hariri & Akylas (1985), is based on an expansion, first, for small U_∞/c. In treating the flow as incompressible, however, the validity of the results is further restricted to small ω/cK.†

4. Assessment based on an orienting model

The issue of estimating the relative contributions σ_1 and σ_2 in (32) and (33), i.e. of p_1^0 and τ_{31}^0 relative to p_2^0, is tantamount, when $K^2\nu/\omega \ll 1$, to comparing the magnitude of (41) with that of (37). Even if T_s and T_{23} are comparable, as we believe likely, the result depends on their dependence (i.e. that of their spectra) on wall distance, y.

For example, consider the following simple, suggestive, and conceivably representative model. Suppose the spectrum, say $S_{32}(y, y', \boldsymbol{K}, \omega)$, corresponding to t_{32} is coherent between points at different wall distance and hence given by a product form

$$S_{32}(y, y', \boldsymbol{K}, \omega) = v_*^3\, \delta^3 \phi(y, \boldsymbol{K}, \omega)\, \phi(y', \boldsymbol{K}, \omega), \tag{45}$$

where v_* is the wall friction speed. Further suppose that the dimensionless function ϕ is given for $K \ll \omega/U_\infty$ (with factor $\delta^{\frac{3}{2}}$ adjoined) by

$$\delta^{\frac{3}{2}}\phi(y, 0, \omega) = By^{\frac{3}{2}}\left(\frac{y}{\delta}\right)^{-\gamma}\left(1+c_1\frac{\omega\delta}{v_*}\right)^{-\gamma}\left[1+y\left(\frac{c_0\,\omega}{v_*}+\frac{1}{b_0\,\delta}\right)\right]^{-(\frac{5}{2}-\alpha)}\left(1+\frac{y}{b\delta}\right)^{-\beta}\left[\frac{y^2}{y^2+\mu^2\nu/\omega}\right]^{s}, \tag{46}$$

where B, b_0, b, c_0, c_1, μ, γ, α, β, s are constants and $\frac{3}{2} \geqslant \gamma \geqslant 0$, $s \geqslant 0$, $\beta \geqslant 0$, $B \sim 1$, $b_0 \sim 1$, $b \sim 1$, $c_0 \sim 1$, $e_1 \sim 1$, $\mu \sim 1$.

The factor $[\;]^{-(\frac{5}{2}-\alpha)}$ in (46) incorporates the possible role of a correlation scale for the fluctuations with reciprocal $\sim y^{-1}+\delta^{-1}$ (since the effective wavenumber is $\sim \omega/v_*$), so that the scale varies as y for $y \ll \delta$ and becomes $\sim \delta$ for $y \gtrsim \delta$. If $\gamma \neq 0$, the factor $(1+c_1\,\omega\delta/v_*)^{-\gamma}$ in (46) implies a role also for a scale $\sim \delta$ independently of y, as further discussed below.

The final factor in (46) is introduced on account of the possible implication of the vanishing of fluctuating-velocity amplitudes as y becomes small relative to $(\nu/\omega)^{\frac{1}{2}}$. The Fourier–Stieltjes amplitude underlying the source density S_{32}, it is recalled, represents a convolution of these amplitudes as in (43). On account of the (three-fold) integration involved in this convolution, however, it appears possible that the convolution may vanish more weakly as $y(\omega/\nu)^{\frac{1}{2}} \to 0$ than does the associated velocity product in (43). Furthermore, part of this effect driving the spectrum to zero as $y \to 0$ may be supplied already by the factor $y^{3-2\gamma}$ that occurs in S_{32} by (45) and (46) in the absence of the final factor.

In the near-wall region where $y \ll \delta$, for $y \gg (\nu/\omega)^{\frac{1}{2}}$ (i.e. with omission of the inner-scale dependent final factor in (46)), one has

$$\delta^{\frac{3}{2}}\phi(y, 0, \omega) \to By^{\frac{3}{2}}(y/\delta)^{-\gamma}(1+c_1\,\omega\delta/v_*)^{-\gamma}(1+c_0\,\omega y/v_*)^{-(\frac{5}{2}-\alpha)}. \tag{47a}$$

† A subsequent treatment, to be reported separately, has been carried out without restriction on ω/cK (but still for small U_∞/c). In a certain approximation that everywhere neglects $\omega\nu/c^2$, the contribution to shear-stress amplitude represented by (40) remains unchanged while the contribution to pressure amplitude of (41) contains an added factor $(1-\omega^2/c^2K^2)^{-\frac{1}{2}}$.

In further sublimits this form becomes

$$\delta^{\frac{3}{2}}\phi(y,0,\omega) \to \begin{cases} By^{\frac{3}{2}}(y/\delta)^{-\gamma} \quad \text{for} \quad \omega\delta/v_* \ll 1, & (47b) \\ By^{\frac{3}{2}}\left(\dfrac{c_1\,\omega y}{v_*}\right)^{-\gamma}\left(1+c_0\dfrac{\omega y}{v_*}\right)^{-(\frac{3}{2}-\alpha)} \quad \text{for} \quad \dfrac{\omega\delta}{v_*} \gg 1. & (47c) \end{cases}$$

According to (47*c*) and (45), in this near-wall region S_{32} becomes independent of δ, i.e. exhibits 'wall similarity', in the domain where, in addition, the effective wavenumber ω/v_* of the fluctuations is large compared with the reciprocal outer scale δ^{-1}. On the other hand, total wall similarity in the near-wall region, meaning scale-independence for *all* $\omega\delta/v_*$ is exhibited by the present model form, in view of (47*b*), only if $\gamma = 0$.

With this characterization of the source model in the background, the resulting *rotational* contribution, say $P_r(0,\omega)$, to the wavevector–frequency spectral density of wall pressure of $\boldsymbol{K}$-aligned shear stress may be obtained from the approximations (40) and (41). These yield

$$P_r(0,\omega) = \rho^2 \int_0^\infty \mathrm{d}\zeta \int_0^\infty \mathrm{d}\zeta' \exp\left[-\frac{(\zeta+\zeta')}{2^{\frac{1}{2}}}\right]\exp\left[\mathrm{i}\frac{(\zeta'-\zeta)}{2^{\frac{1}{2}}}\right] S_{32}\left(\left(\frac{\nu}{\omega}\right)^{\frac{1}{2}}\zeta, \left(\frac{\nu}{\omega}\right)^{\frac{1}{2}}\zeta', 0, \omega\right). \tag{48}$$

For the coherent model (45), this becomes simply

$$P_r(0,\omega) = \rho^2 v_*^3\,\delta^3 |I_0|^2, \tag{49a}$$

$$I_0 \equiv \int_0^\infty \mathrm{d}\zeta \exp(\mathrm{i}^{\frac{3}{2}}\zeta)\,\phi\left(\left(\frac{\nu}{\omega}\right)^{\frac{1}{2}}\zeta, 0, \omega\right). \tag{49b}$$

By estimating the integral (49*b*) in limiting domains one can arrive at a rough result for P_r of (49*a*). Consider, in particular, the (inviscid) subdomain where $\omega\nu/v_*^2 \lesssim 1$ and $\omega\delta/v_* \gtrsim 1$. With these restrictions, (49) yields

$$P_r(0,\omega) \sim \rho^2 v_*^6\,\omega^{-3}(\omega\nu/v_*^2)^{\frac{3}{2}-\gamma}. \tag{50}$$

If $\gamma = \frac{3}{2}$, estimate (50) reduces simply to the scale-independent, wavevector-white form $\sim \rho^2 v_*^6\,\omega^{-3}$ that is most commonly accepted as describing turbulent wall pressure in the range of subconvective (but incompressive) wavenumbers under discussion here. Moreover, it appears, the level coefficient associated with this rotational contribution may suffice to account for the roughly measured level (although Hariri & Akylas (1985), on the basis, apparently, of a less specific appraisal, seem to have concluded otherwise). This measured level, when normalized such that mean squared pressure is obtained by integration over the doubly infinite range of radian frequency, lies on the order of 10 or 20 dB below that of $\rho^2 v_*^6\,\omega^{-3}$.

A value $\gamma = \frac{3}{2}$ corresponds according to (45) and (47*b*) to a source spectrum $S_{32}(y,y,0,\omega)$ possessing incomplete wall similarity in the sense noted above, varying as $v_*^3\,\delta^3$ in the near-wall (but inviscid) region in the low-frequency limit, rather than as $v_*^3\,y^3$. Whether such a source model is sustainable in the light of known and presumptive properties of the boundary-layer fluctuations warrants careful consideration. It should be further noted that, even if the choice $\gamma = 0$ in (46) were required of a tenable model, as $\omega\nu/v_*^2$ increases the estimated functional form of $P_r(0,\omega)$ decreases relative to the result (50) obtained for sufficiently small $\omega\nu/v_*^2$, so that near scale-independence may prevail over the range of a flat peak possibly encompassing the usual domain of measurement.

If the rotational contribution, in fact, dominates wall pressure (as well as shear stress) in the low-wavenumber domain discussed here, the Kraichnan–Phillips theorem, which is based on an inviscid approximation, is rendered moot, and the present lack of experimental substantiation of it (Chase 1990) may be understood. Likewise, source-based modelling of the irrotational contribution, say $P_i(\boldsymbol{K}, \omega)$, is freed from the constraint of accounting for the measured properties of the low-wavenumber domain (cf. Chase 1987). This contribution (obtained from (37) with the modification cited), then, might well have roughly the dependence (omitting directionality)

$$P_i(\boldsymbol{K}, \omega) \sim \rho^2 v_*^6\, \omega^{-3} (v_* K/\omega)^2,$$

once suggested as most likely (Chase 1980).

The present results also support the necessity for considering fluctuating wall-shear stress along with pressure in various applications.

Stimulating discussions with Professor R. E. Kronauer and Dr M. S. Howe are gratefully acknowledged. This work has been supported by the US Navy Office of Naval Research, Mechanics Division, Code 1132F, Fluid Dynamics, Scientific Officer: Dr M. M. Reischman, under the Solid–Fluid Interaction Accelerated Research Initiative, Contract no. N00014-88-C-0673.

REFERENCES

Bender, C. M. & Orszag, S. A. 1978 *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill.

Bergeron, R. F. 1973 Aerodynamic sound and the low-wavenumber wall-pressure spectrum of nearly incompressible boundary layer turbulence. *J. Acoust. Soc. Am.* **54**, 123–133.

Chase, D. M. 1980 Modeling the wavevector–frequency spectrum of turbulent-boundary-layer wall pressure. *J. Sound Vib.* **70**, 29–57.

Chase, D. M. 1987 The character of the turbulent wall pressure spectrum at subconvective wavenumbers and a suggested comprehensive model. *J. Sound Vib.* **112**, 125–147.

Chase, D. M. 1990 The wavevector–frequency spectrum of pressure on a smooth plane in turbulent boundary-layer flow at low Mach number. *J. Acoust. Soc. Am.* (in press).

Chase, D. M. & Noiseux, C. F. 1982 Turbulent wall pressure at low wavenumbers: relation to nonlinear sources in planar and cylindrical flow. *J. Acoust. Soc. Am.* **72**, 975–982.

Hariri, H. H. & Akylas, T. R. 1985 The wall-shear-stress contribution to boundary-layer noise. *Phys. Fluids* **28**, 2727–2729.

Howe, M. S. 1989 Turbulent boundary layer shear stress fluctuations on smooth and rough walls. *BBN Systems and Technologies Corp. TM* 1025.

Kraichnan, R. H. 1956 Pressure fluctuations in turbulent flow over a flat plate. *J. Acoust. Soc. Am.* **28**, 278–390.

Phillips, O. M. 1956 On the aerodynamic surface sound from a plane turbulent boundary layer. *Proc. R. Soc. Lond.* A **234**, 327–335.